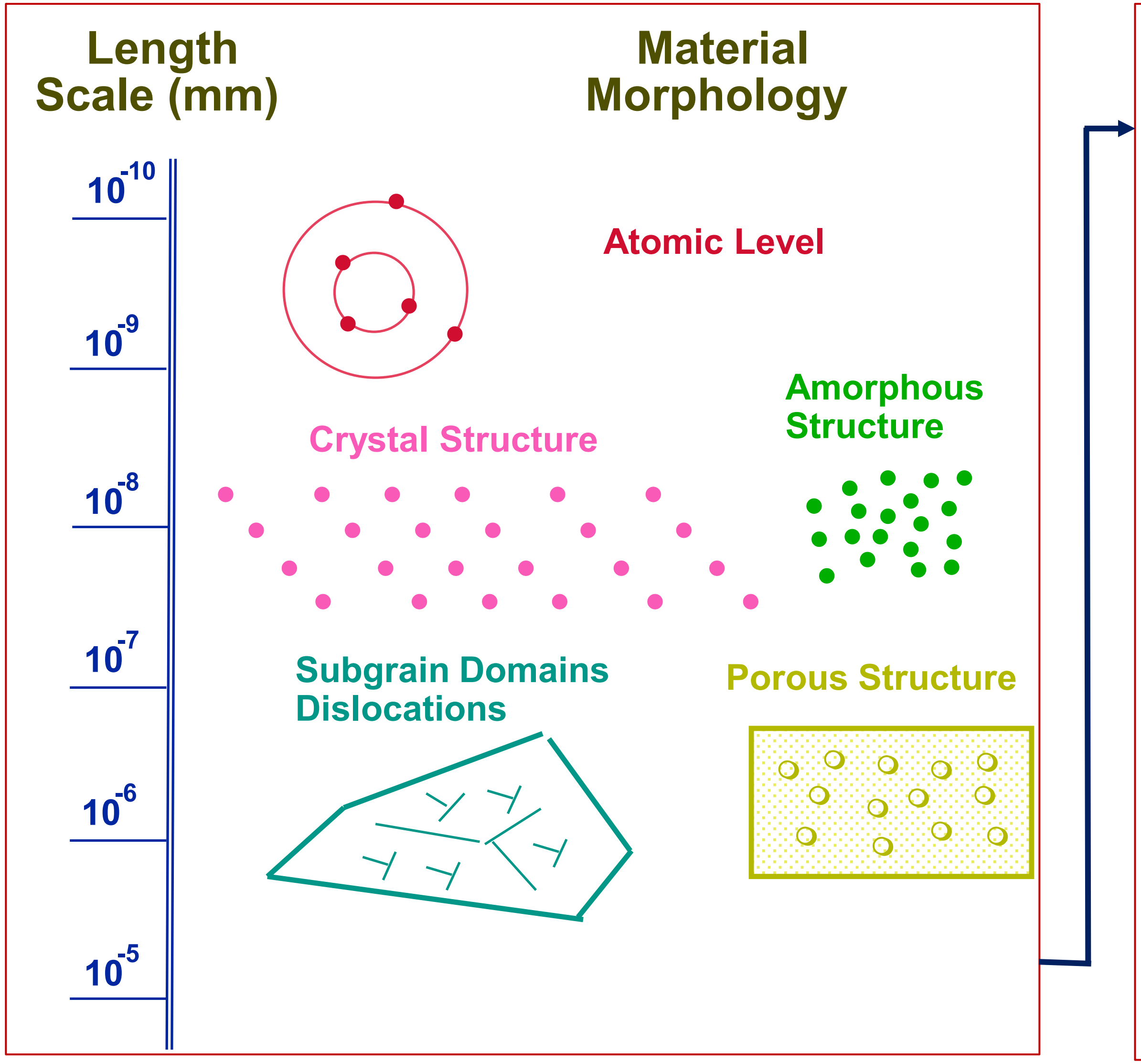
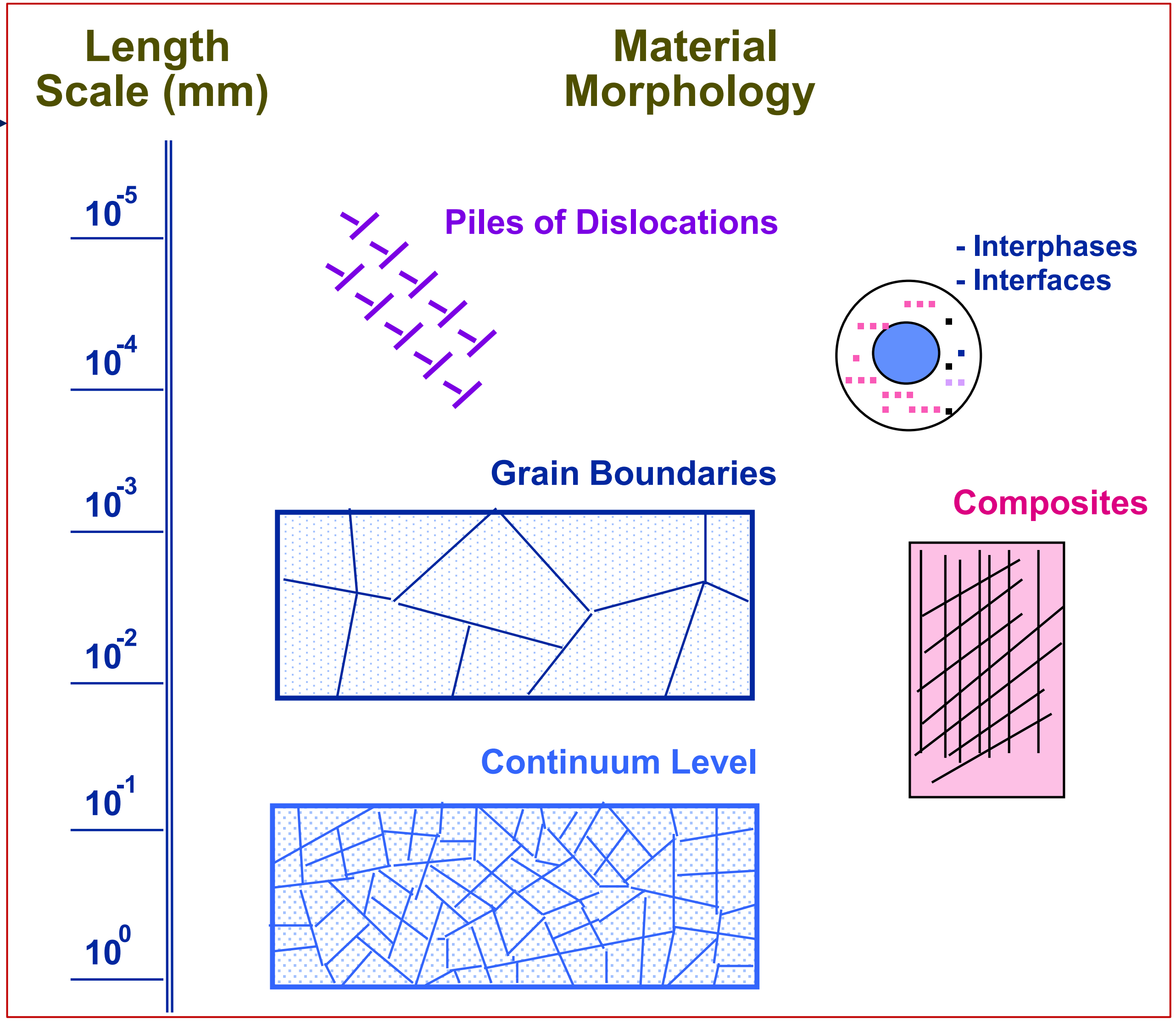
Index notation and Cartesian Tensors

- Index notation, Summation convention
- Orthogonal Transformation
- Scalars, Vectors, Tensors
- Tensor Properties and Operations
- Gauss theorem

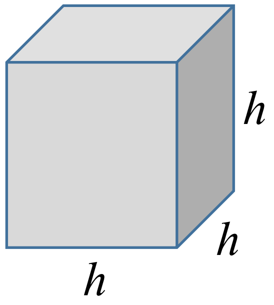
From the book: Mechanics of Continuous Media: an Introduction
J Botsis and M Deville, PPUR 2018.
Solutions: https://www.epflpress.org/produit/908/9782889152810/mechanics-of-continuous-media

Continuum mechanics review (material length scales)





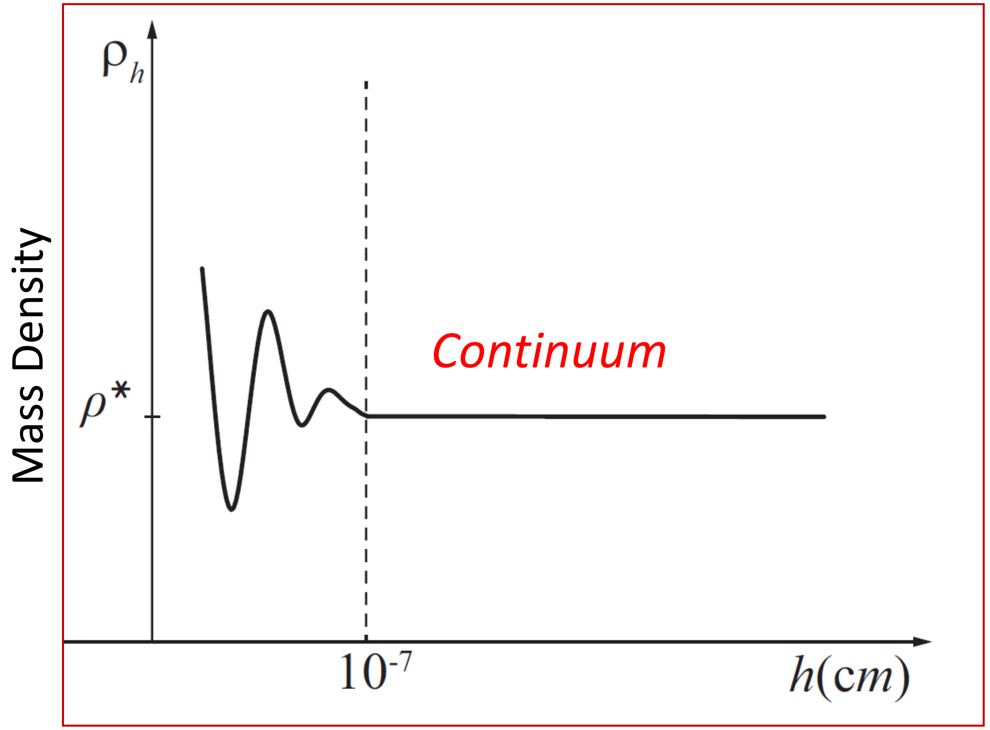
Continuum mechanics review



Mass
Density

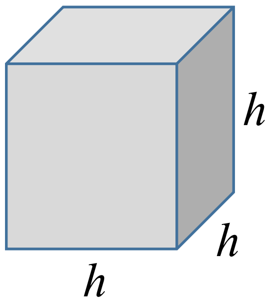
$$\rho_h = \frac{M}{h^3}$$

Experimental Evidence (Fluid)



We ignore the discrete aspect of matter and we consider that properties such as density, viscosity, modulus of elasticity, etc., assigned to a midpoint continuous are continuous functions of space variables.

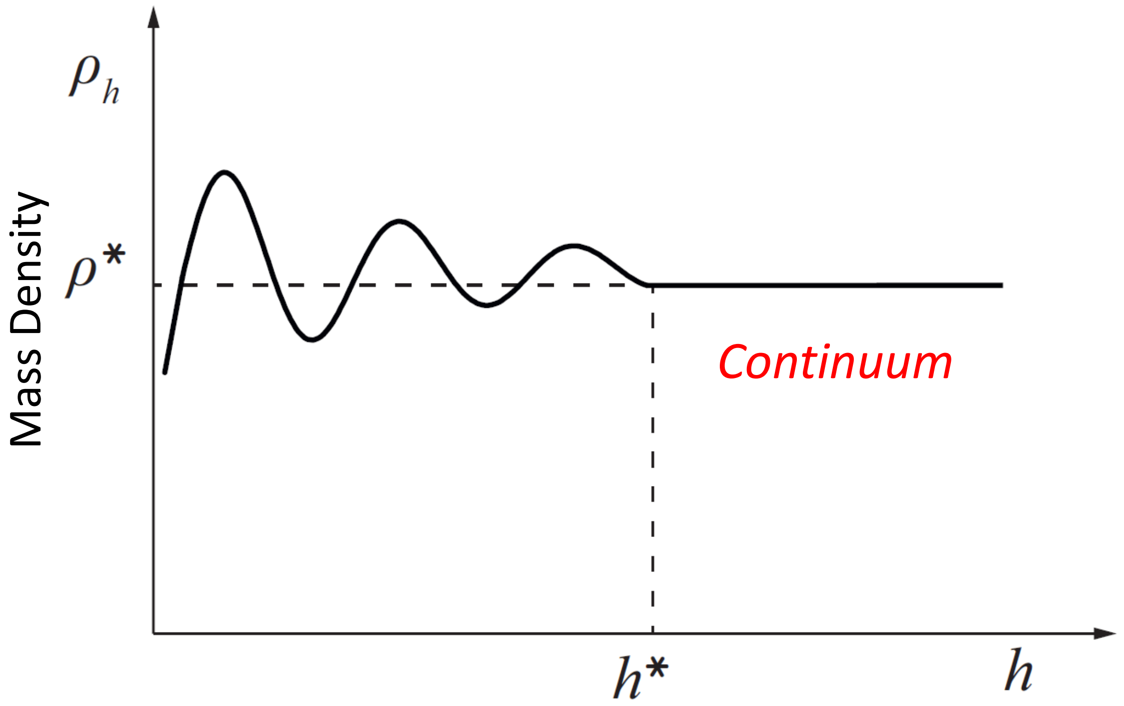
Continuum mechanics review



Mass Density

$$\rho_h = \frac{M}{h^3}$$

Experimental Evidence (Solid)



We ignore the discrete aspect of matter and we consider that properties such as density, viscosity, modulus of elasticity, etc., assigned to a midpoint continuous are continuous functions of space variables.

Continuum mechanics review-Vector Algebra

In mechanics of continuous media, motion and the associated physical quantities are described in ***Euclidean space*** $\mathbf{R}^3$ (physical space) with which a three-dimensional vector space E^3 is associated.

The elements of $\mathbf{R}^3$ and E^3 are called the points and vectors, respectively.

The scalars, vectors, and tensors that describe the physical quantities are also attached to a space (typically $\mathbf{R}^3$) and form what are called scalar, vector, or tensor fields.

Continuum mechanics review-Vector Algebra

A ***vector space*** is defined uniquely from the properties of operations on its elements and assumes the existence of an arbitrary field (typically the field of real numbers **R**) whose elements are called scalars. The ***vector space*** E^3 is then the set of elements denoted $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \ldots$ such that:

$$\begin{aligned}
&\boldsymbol{u}+\boldsymbol{v} \in E^3 && a\,\boldsymbol{u} \in E^3 \\
&(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w} = \boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w}) && 1\boldsymbol{u} = \boldsymbol{u} \\
&\exists \boldsymbol{0} \in E^3 \mid \boldsymbol{u}+\boldsymbol{0} = \boldsymbol{u} && a(b\boldsymbol{u}) = (ab)\boldsymbol{u} \qquad (1.1) \\
&\exists -\boldsymbol{u} \in E^3 \mid \boldsymbol{u}+(-\boldsymbol{u}) = \boldsymbol{0} && (a+b)\boldsymbol{u} = a\boldsymbol{u}+b\boldsymbol{u} \\
&\boldsymbol{u}+\boldsymbol{v} = \boldsymbol{v}+\boldsymbol{u} && a(\boldsymbol{u}+\boldsymbol{v}) = a\boldsymbol{u}+a\boldsymbol{v}
\end{aligned}$$

for every $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in E^3$ and $a, b \in$ **R**.

By providing E^3 with a scalar product, to be able to calculate lengths and angles, it takes the name Euclidean space. The scalar product associates with every pair of vectors $\boldsymbol{u}, \boldsymbol{v} \in E^3$ a scalar denoted $\boldsymbol{u}\cdot\boldsymbol{v}$ with the following properties:

$$\begin{aligned}
&\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{u} \\
&\boldsymbol{u}\cdot(\alpha\boldsymbol{v}+\beta\boldsymbol{w}) = \alpha(\boldsymbol{u}\cdot\boldsymbol{v})+\beta(\boldsymbol{u}\cdot\boldsymbol{w}) \qquad (1.2) \\
&\boldsymbol{u}\cdot\boldsymbol{u} \geq 0
\end{aligned}$$

for every $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in E^3$ and $\alpha, \beta \in$ **R**.

Note: numbers refer to the equations in the book by Botsis & Deville

Continuum mechanics review-Vector Algebra

The ***scalar product*** is consequently an application of E^3 x E^3 in **R** that is linear with respect to each of its arguments. It is also called a positive definite bilinear form (See (1.2) in the previous slide).

The definition of the ***vector norm*** $\boldsymbol{u}$, denoted $\|\boldsymbol{u}\|$ is given by the relation:

$$\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}\,. \qquad (1.3)$$

The vector $\boldsymbol{u}$ is called a unit vector when $\|\boldsymbol{u}\|$ = 1, and two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal if and only if $\boldsymbol{u} \cdot \boldsymbol{v}$ = 0.

Every vector in E^3 can be decomposed uniquely according to a basis formed of three linearly independent vectors of E^3. The choice of a basis is arbitrary but generally one uses the ***canonical basis*** ($\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$) defined by:

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j\,, \end{cases} \quad i, j = 1, 2, 3. \qquad (1.4)$$

The basis is called ***orthogonal*** when the basis vectors are not unit vectors but are still orthogonal.

Continuum mechanics review-Vector Algebra

Physical quantities in continuum mechanics

Scalars: Quantities for which only one value can be associated. For example, the mass ***density*** of a material. We denote it ρ and it has for SI units kg/m^3 or dimensions ML^{-3} where M is the mass and L a length. This density is practically constant and, in addition, there is no direction associated with its value.

Vectors: quantities have not only a value but also a direction.
A force of one Newton is that which, applied to a point, gives it an ***acceleration*** of 1 m s^{-2} per kg.
Since this force has a direction, it is a vector. In a given coordinate system, this vector is specified by its components. Going from one set of axes to another, the vector remains invariant and only the components of the vector change by a transformation rule.

Tensors: We introduce the concept of a tensor in a simplistic way as follows. Consider, a stress is a force per unit surface: (a) force is a vector and (b) an element of a surface is also a vector (must specify both its size and its orientation).
If ***f*** describes the force vector and ***s*** the normal vector of the surface *S*, then we might think that the stress ***T*** could be expressed by ***f*** /***s***. But, as the division of two vectors is an undefined operation, we get around the difficulty by saying that given ***s***, we can find ***f*** by multiplying ***s*** by a new entity ***T*** such that:

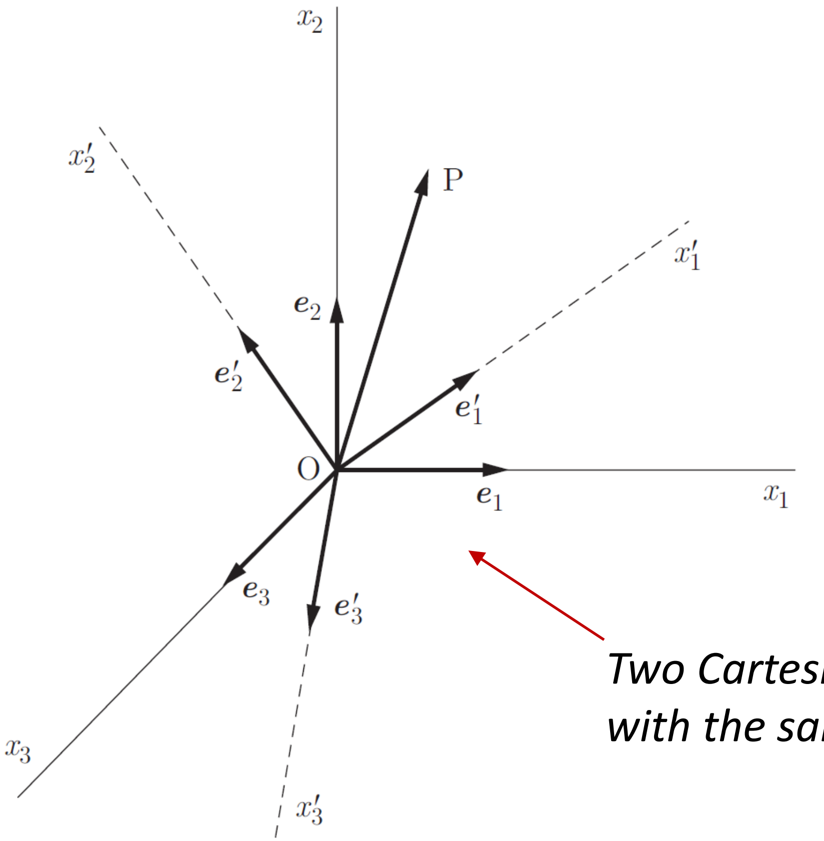
$$\boldsymbol{f}(\boldsymbol{s}) = \boldsymbol{T}\boldsymbol{s}$$

This mathematical object is a tensor which yields the stress at a point and is a tensor of order 2. **It is associated with two spatial directions** and can be represented by a matrix with two indices, each index corresponding to one direction in Euclidean space. It is thus an entity with nine components.

Continuum mechanics review-Orthogonal Transformation

In the physical Euclidean space $\mathbf{R}^3$, let there be a Cartesian orthonormal co- ordinate system (O, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$), that we denote also as Ox_i (i = 1, 2, 3), with origin at O and the unit vectors $\boldsymbol{e}_i$ (i = 1, 2, 3) directed along the axes Ox_i (fig. 1.3).

Another system $O\,x_i'$ (i = 1, 2, 3) with unit vectors $\boldsymbol{e}_i'$ defines a Cartesian coordinate system with the same origin O. The ***direction cosines*** of the axes x_i' with respect to the axes x_i, denoted by c_{pi}, are given by the scalar products of the basis vectors.



$$c_{pi} = \cos(x_p', x_i) = \boldsymbol{e}_p' \cdot \boldsymbol{e}_i \qquad i, p = 1, 2, 3\,. \tag{1.6}$$

Similarly, the direction cosines of the first system with respect to the second are given by:

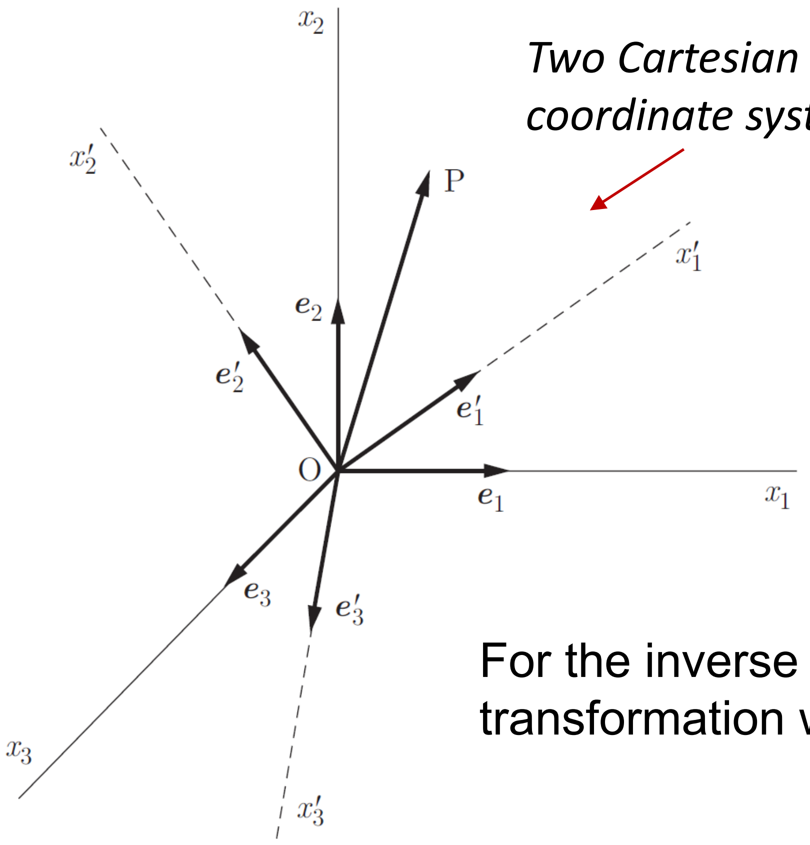
$$c_{pi}' = \boldsymbol{e}_p \cdot \boldsymbol{e}_i' = c_{ip}\,, \tag{1.7}$$

Two Cartesian coordinate systems with the same origin

Continuum mechanics review-Orthogonal Transformation

Let P be a point with coordinates x_i in the first system and x_i' in the second.

From equation (1.6), the coordinates x_i' are related to those of x_i and x_i'



Two Cartesian coordinate systems

Using equation (1.6) $c_{pi} = \cos(x_p', x_i) = \mathbf{e}_p' \cdot \mathbf{e}_i$

$$\begin{aligned} x_1' &= c_{11}x_1 + c_{12}x_2 + c_{13}x_3 \\ x_2' &= c_{21}x_1 + c_{22}x_2 + c_{23}x_3 \\ x_3' &= c_{31}x_1 + c_{32}x_2 + c_{33}x_3 \,. \end{aligned} \tag{1.8}$$

Or

$$x_i' = \sum_{j=1}^{3} c_{ij}x_j \qquad i = 1, 2, 3\,. \tag{1.9}$$

For the inverse transformation we have:

$$\begin{aligned} x_1 &= c_{11}x_1' + c_{21}x_2' + c_{31}x_3' \\ x_2 &= c_{12}x_1' + c_{22}x_2' + c_{32}x_3' \\ x_3 &= c_{13}x_1' + c_{23}x_2' + c_{33}x_3' \end{aligned} \tag{1.10}$$

Or

$$x_i = \sum_{j=1}^{3} c_{ji}x_j' = \sum_{j=1}^{3} c_{ij}'x_j'\,. \tag{1.11}$$

Summation convention

We can suppress the symbol Σ by adopting, from here on, the Einstein ***summation convention*** for repeated indices to write for the coordinates of point P and agree to the following:

When an index appears twice in a product, a sum with respect to that index is implied by taking successively all its possible values (in this case, i = 1, 2, 3). In this way equations (1.9) and (1.11),

$$x'_i = \sum_{j=1}^{3} c_{ij} x_j \qquad i = 1, 2, 3 . \tag{1.9}$$

$$x_i = \sum_{j=1}^{3} c_{ji} x'_j = \sum_{j=1}^{3} c'_{ij} x'_j . \tag{1.11}$$

are written in the compact form:

$$x'_i = c_{ij} x_j \qquad x_i = c_{ji} x'_j \qquad j = 1, 2, 3 . \tag{1.12}$$

Summation convention

Illustrations of ***summation convention***

$$\sigma_{ij}n_j = \sum_{j=1}^{3} \sigma_{ij}n_j = \sigma_{i1}n_1 + \sigma_{i2}n_2 + \sigma_{i3}n_3$$

(the index *i* is fixed and has a value among 1, 2, 3. It's called **free index**. We sum over the repeated index).

$$\sigma_{ij}n_jn_i = \sum_{j=1}^{3}\sum_{i=1}^{3} \sigma_{ij}n_jn_i = \sigma_{11}n_1^2 + \sigma_{22}n_2^2 + \sigma_{33}n_3^2 + (\sigma_{12}+\sigma_{21})n_1n_2 + (\sigma_{23}+\sigma_{32})n_2n_3 + (\sigma_{31}+\sigma_{13})n_3n_1 \, .$$

(No free index. We sum over the two repeated indices *i and j* =1, 2, 3)

$$L_{ii} = \sum_{i=1}^{3} L_{ii} = L_{11} + L_{22} + L_{33}$$

$$A_iB_kC_i = \sum_{i=1}^{3} A_iB_kC_i = B_k \sum_{i=1}^{3} A_iC_i = B_k(A_1C_1 + A_2C_2 + A_3C_3)$$

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 = \sum_{i=1}^{3} dx_i\,dx_i = dx_i\,dx_i \, .$$

u_i designated the set of the 3^1 quantities u_1, u_2, u_3 (3 for space and 1 for the free index).
L_{ij} signifies the set of the 3^2 quantities L_{11}, L_{12}, L_{13}, L_{21}, L_{22}, L_{23}, L_{31}, L_{32}, L_{33} (3 for space and 2 for the free indices).
L_{ii}, we have $3^0 = 1$ quantity.

Summation convention and Kronecker delta

Illustrations of ***summation convention***

Note that the index over which we sum is a dummy index;
we can change the notation of this index without changing the significance of the sum:

$$\sigma_{ij}n_j = \sigma_{ik}n_k = \sigma_{il}n_l$$

$$M_{ijk}u_iv_jw_k = M_{jik}u_jv_iw_k = M_{ikj}u_iv_kw_j = \dots$$

Very important in operations with complex equations

A dummy index does not appear more than twice in a product.
Using this property and

$$x'_i = c_{ij}x_j \qquad x_i = c_{ji}x'_j \qquad (1.12)$$

(2) (1)

we can wrirte: $x_j = c_{qj}x'_q$

(3)

from which: $x'_i = c_{ij}c_{qj}x'_q$

$$x_i = c_{ji}c_{jq}x_q$$

The coefficients in the last equations should be equal to one when i=q and 0 for i not q. This is also true for the other equation.

We can introduce the **Kronecker delta** to reflect such a property.

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j\,, \end{cases} \qquad (1.14)$$

$$c_{ij}c_{qj} = \delta_{iq}\,. \qquad (1.15)$$

Summation convention, Kronecker delta

Orthogonality conditions: From the Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j\,, \end{cases} \qquad (1.14)$$

and $x'_i = c_{ij} c_{qj} x'_q$

we can wrirte the orthogonality condition:

$$c_{ij} c_{qj} = \delta_{iq}$$

The components c_{ij} form an orthogonal matrix $[C]$ such that:

$$[C][C]^{-1} = [C][C]^T = [I]\,, \qquad (1.16)$$

with

$$\det[C] = \boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3)\,, \qquad (1.17)$$

$\det[C] = \pm 1$((+) indicates a direct rotation and (-) a reflection)

The Kronecker delta can be used to change the index of a component:

(very useful property)

$$\begin{aligned} L_{ik} &= \delta_{ij} L_{jk}, \\ A_i B_k C_i &= \delta_{ij} A_i B_k C_j, \\ \frac{\partial u_j}{\partial x_i} &= u_{j,i} = \delta_{kj} \frac{\partial u_k}{\partial x_i} = \delta_{kj} u_{k,i} \\ \frac{\partial^2 u_i}{\partial x_j \partial x_k} &= u_{i,jk} = \delta_{jl} \frac{\partial^2 u_i}{\partial x_l \partial x_k} = \delta_{jl} u_{i,lk}\,. \end{aligned}$$

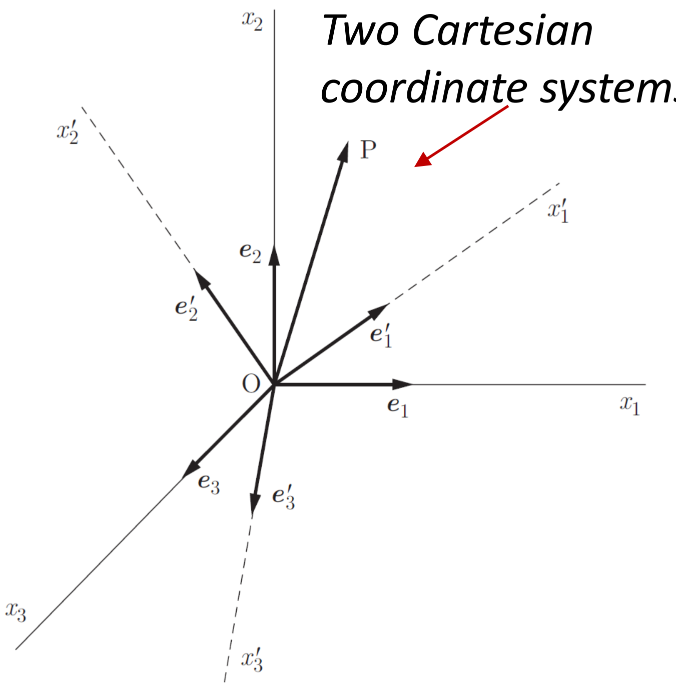
Note also the following relations usefull in deriving relations in mechanics:

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} \quad ; \quad \frac{\partial A_{ik}}{\partial A_{jl}} = \delta_{ij} \delta_{kl}$$

Scalars

Recall: In mechanics of continuous media, we work with scalars, vectors, and tensors that describe the physical quantities that are also attached to a space (typically **R**^{3}) and form what are called scalar, vector, or tensor fields. **What is important in the context of continuous media is how we can describe these quantities in different coordinate systems.**

Consider a point P in a continuous medium and a real value continuous function F (P) at P.
If the value F (P) does not depend on the coordinate system, then F is called a scalar function, or scalar, or a tensor of order 0. This is the case, for example, for temperature, pressure, kinetic energy, etc.



Two Cartesian coordinate systems

Suppose that P has coordinates x_i and if F (P) has a value $f(x_i)$, the change of coordinate systems:

$$x_i = c_{ji} x'_j$$

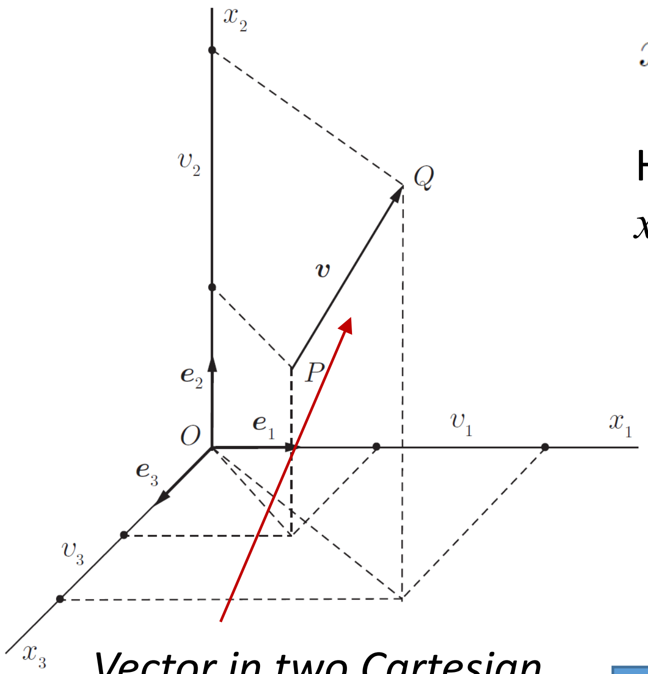
for the scalar F (P) leads to

$$F(P) = f(x_i) = f(c_{ji} x'_j) = f'(x'_j)$$

Note that the values remains the same but the form of the function can varie in the new coordinate system.

Vectors

Let $\boldsymbol{v}$ = ***P Q*** be a vector having its origin at point P and its extremity at point Q (fig. 1.4). This vector has a direction and three components v_i. The vector itself is independent of the coordinate system.



Vector in two Cartesian coordinate systems

$$x'_i = c_{ij} x_j \quad \text{and} \quad x_i = c_{ji} x'_j \,. \qquad (1.19)$$

Here x_i, y_i are the coordinates of the points P and Q in the first coordinate system and x_i', y_i' in the second. The components of $\boldsymbol{v}$ in the two systems are written as:

$$v_i = y_i - x_i$$
$$v'_i = y'_i - x'_i$$

(1.19) →

$$v'_i = y'_i - x'_i = c_{ij}(y_j - x_j) = c_{ij} v_j \quad (1.20)$$

In a Cartesian coordinate system the c_{ij} are Independent of the coordinates of P.

(1.19) →

$$\frac{\partial x'_i}{\partial x_j} = c_{ij} \qquad \frac{\partial x_i}{\partial x'_j} = c_{ji} \quad (1.21)$$

→

$$\frac{\partial x_i}{\partial x'_j} = \frac{\partial x'_j}{\partial x_i}$$

(1.20), (1.21) →

$$v'_i = \frac{\partial x'_i}{\partial x_j} v_j \quad \text{or} \quad v'_i = \frac{\partial x_j}{\partial x'_i} v_j$$

(1.22)

The object $\boldsymbol{v}$, characterized by the three components v_i in a Cartesian coordinate system, is a vector or a tensor of order 1 if its components are transformed according to this rule during a coordinate system change that is an orthogonal transformation satisfying (1.16).

Vectors

Using index notation we can express the algebra of vectors and their components:

With a being a scalar we can write, using (1.22):

$$(av_i') = c_{ij}(av_j) = \frac{\partial x_i'}{\partial x_j}(av_j)$$

the **vector addition** results in a vector with components:

$$w_i = u_i + v_i$$

or in vector notation : $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$

With $\|\boldsymbol{u}\|$ and $\|\boldsymbol{v}\|$ being the norms of the two vectors we have :

$$b = \|\boldsymbol{u}\|\,\|\boldsymbol{v}\|\cos\theta$$

θ is the angle between the two vectors).

For a single vector: $v_i v_i = \|\boldsymbol{v}\|^2$

Scalar product of two vectors b:

In symbolic form : $b = \boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$

In index form : $b = u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3$

The scalar product of two vectors is independent of an orthogonal change of coordinates.

Using (1.20), we have:

$$v_i = y_i - x_i = c_{ji}(y_j' - x_j') = c_{ji} v_j'$$

$$b = u_i v_i = c_{ji} u_j' c_{ki} v_k' = c_{ji} c_{ki} u_j' v_k'$$

$$= \delta_{jk} u_j' v_k' = u_j' v_j'$$

Permutation symbol and vector product

The permutation symbol is defined as:

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of } 1\,2\,3 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 1\,2\,3 \\ 0 & \text{all other cases} \end{cases}$$

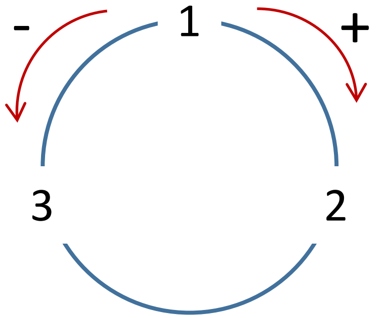
or

$$\varepsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)$$

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$$

$$\varepsilon_{ijk} = -\varepsilon_{jik}$$

$$\varepsilon_{ijk} = -\varepsilon_{ikj}$$



Permutation symbol and vector product

With the **Kronecker** symbol

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

and the permutation symbol we have
an important relation used to easily demonstrate vector identities
and express vector products in index form.

In an orthonormal basis of **R**^{3}, **the vector product of two vectors** $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v}$, sometimes denoted $\boldsymbol{u} \wedge \boldsymbol{v}$, is defined by the equality:

$$w_i = \varepsilon_{ijk} u_j v_k$$

The norm of the vector is given by the equality
(with θ angle between $\boldsymbol{u}$ and $\boldsymbol{v}$):

$$||\boldsymbol{w}|| = ||\boldsymbol{u}|| \, ||\boldsymbol{v}|| \sin\theta$$

Permutation symbol and vector product

Example 1.1

The vector product $\boldsymbol{u} \times \boldsymbol{v}$ generates a vector $\boldsymbol{w}$ perpendicular to the plane of the two vectors, and the three vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ form a direct system. It can be shown that $\boldsymbol{w}$ is orthogonal to $\boldsymbol{v}$ since $\boldsymbol{v} \cdot \boldsymbol{w}$ is zero.

$$
\begin{aligned}
v_i w_i &= v_i \varepsilon_{ijk} u_j v_k = \varepsilon_{ijk} v_i v_k u_j \\
&= \frac{1}{2}\left(\varepsilon_{ijk} v_i v_k u_j + \varepsilon_{ijk} v_i v_k u_j\right) = \frac{1}{2}\left(\varepsilon_{ijk} v_i v_k u_j + \varepsilon_{kji} v_k v_i u_j\right) \\
&= \frac{1}{2}\left(\varepsilon_{ijk} v_i v_k u_j - \varepsilon_{ijk} v_i v_k u_j\right) = 0\,.
\end{aligned}
$$

Permutation symbol and vector product

Example 1.2: Verify the following identity:

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}) \quad (1.36)$$

We use index notation algebra to verify the identity (1.36). The term on the left $\mathfrak{L}$ is written as

$$\mathfrak{L} = (\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = \varepsilon_{ijk} a_j b_k \varepsilon_{ilm} c_l d_m \, .$$

With (1.30), we obtain

$$\begin{aligned} \mathfrak{L} &= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) \, a_j b_k c_l d_m \\ &= \delta_{jl} a_j c_l \delta_{km} b_k d_m - \delta_{jm} a_j d_m \delta_{kl} b_k c_l \, . \end{aligned}$$

Using the properties of the Kronecker delta δ_{ij}, we set $l = j$ and $m = k$ in the first term and $m = j$ and $l = k$ in the second. Then

$$\mathfrak{L} = a_j c_j b_k d_k - a_j d_j b_k c_k \, .$$

The right-hand term of this relation is none other than the index notation representation of the right-hand term of (1.36).

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

(1.30)

Tensor Algebra: definition of a tensor

Definition of a tensor of order 2:

Let E^3 be the Euclidean vector space of vectors associated with $\mathbf{R}^3$, and $\boldsymbol{L}$ a linear mapping on E^3 that transforms a vector to another:

$\boldsymbol{L} : E^3 \rightarrow E^3$ such that: $\boldsymbol{u} \mapsto \boldsymbol{L}\boldsymbol{u}$

If $\boldsymbol{L}$ transforms the two arbitrary vectors as $\boldsymbol{L}\boldsymbol{u}_1 = \boldsymbol{v}_1 \quad \boldsymbol{L}\boldsymbol{u}_2 = \boldsymbol{v}_2$

and has the properties

$$\boldsymbol{L}(\boldsymbol{u}_1 + \boldsymbol{u}_2) = \boldsymbol{L}\boldsymbol{u}_1 + \boldsymbol{L}\boldsymbol{u}_2$$
$$\boldsymbol{L}(\alpha \boldsymbol{u}_1) = \alpha \boldsymbol{L}\boldsymbol{u}_1 \ , \qquad (1.38)$$

where $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two arbitrary vectors of E^3 and $\alpha \in \mathbf{R}$, then we say that $\boldsymbol{L}$ is a linear transformation.
<u>It is also a tensor of order 2, or simply, a tensor</u>.

The ***unit tensor*** $\boldsymbol{I}$ and the ***zero tensor*** $\boldsymbol{O}$ are defined by the relations $\boldsymbol{u} = \boldsymbol{I}\boldsymbol{u}$ and $\mathbf{0} = \boldsymbol{O}\boldsymbol{u}$.

Tensor Algebra: definition of a second order tensor

For a vector $\boldsymbol{u}$, the vector $\boldsymbol{v}$ is given by: $\boldsymbol{v} = \boldsymbol{L}\boldsymbol{u} = \boldsymbol{L}u_i\boldsymbol{e}_i = u_i\boldsymbol{L}\boldsymbol{e}_i$ (1.39)

If we express the components of $\boldsymbol{v}$ as: $v_i = \boldsymbol{e}_i \cdot \boldsymbol{v}$ (1.40)

$$\Rightarrow \quad v_i = \boldsymbol{e}_i \cdot (u_j\boldsymbol{L}\boldsymbol{e}_j) = u_j\boldsymbol{e}_i \cdot \boldsymbol{L}\boldsymbol{e}_j \quad (1.41)$$

The components of the tensor are: $L_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{L}\boldsymbol{e}_j$ (1.42)

or $\boxed{v_i = L_{ij}u_j}$ (1.43)

which is the transformation of the two vectors in index form. In a matrix from, it is.

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

From (1.42) we see that the columns of the matrix are the components of vectors $\boldsymbol{L}\boldsymbol{e}_i$ and depend on the coordinate system. However, the operator does not depend on the base vector.

matrix associated with the tensor

$$[L] = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix}$$

determinant associated with the tensor

$$\det \boldsymbol{L} = \det[L] = \det \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix}$$

Tensor Algebra: Dyadic (or tensor) product of two vectors

The tensor product or dyadic product $\boldsymbol{a} \otimes \boldsymbol{b}$ of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as the tensor which, for any vector $\boldsymbol{v}$ defines the following transformation :

$$(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{v} = (\boldsymbol{b} \cdot \boldsymbol{v})\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{v}) \qquad (1.48)$$

For every vector $\boldsymbol{v}$ and $\boldsymbol{w}$ and for $\alpha, \beta \in \mathrm{R}$, we have

$$\begin{aligned}(\boldsymbol{a} \otimes \boldsymbol{b})(\alpha\boldsymbol{v} + \beta\boldsymbol{w}) &= (\boldsymbol{b} \cdot (\alpha\boldsymbol{v} + \beta\boldsymbol{w}))\boldsymbol{a} \\ &= (\alpha(\boldsymbol{b} \cdot \boldsymbol{v}) + \beta(\boldsymbol{b} \cdot \boldsymbol{w}))\boldsymbol{a} \\ &= \alpha(\boldsymbol{b} \cdot \boldsymbol{v})\boldsymbol{a} + \beta(\boldsymbol{b} \cdot \boldsymbol{w})\boldsymbol{a} \\ &= \alpha(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{v} + \beta(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{w}\end{aligned}$$

The definition of the dyadic product and the last relation demonstrates that the product is a tensor. Its components are:

(1.42)

$$(a \otimes b)_{ij} = \boldsymbol{e}_i \cdot (\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{e}_j =$$

$$= \boldsymbol{e}_i \cdot ((\boldsymbol{b} \cdot \boldsymbol{e}_j)\boldsymbol{a}) = \boldsymbol{e}_i \cdot (\boldsymbol{a}b_j) = (\boldsymbol{e}_i \cdot \boldsymbol{a})b_j = a_i b_j$$

The matrix form of a dyadic product is:

$$[\boldsymbol{a} \otimes \boldsymbol{b}] = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix}$$

Tensor Algebra: Dyadic (or tensor) product of two vectors

For two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$,

$$(\boldsymbol{u} \otimes \boldsymbol{v}) \neq (\boldsymbol{v} \otimes \boldsymbol{u})$$

Note that :

$$(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\boldsymbol{u} = (\boldsymbol{e}_j \cdot \boldsymbol{u})\boldsymbol{e}_i = u_j \boldsymbol{e}_i$$

With the help of (1.43) we obtain:

(1.43) →

$$\boldsymbol{v} = v_i \boldsymbol{e}_i = \boldsymbol{L}\boldsymbol{u} = L_{ij} u_j \boldsymbol{e}_i = L_{ij}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\boldsymbol{u}$$

and

$$\boldsymbol{L} = L_{ij}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)$$

$$v_i = L_{ij} u_j \qquad (1.43)$$

Note two important relations:

$$\boldsymbol{I} = \delta_{ij}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)$$

$$\boldsymbol{a} \otimes \boldsymbol{b} = a_i b_j (\boldsymbol{e}_i \otimes \boldsymbol{e}_j)$$

Tensor Algebra: transformation rule for Cartesian tensor

The representation in the Cartesian coordinate system x_i of the linear operator $\boldsymbol{L}$,

which is invariant, is given by its components L_{ij} (1.42) $L_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{L}\boldsymbol{e}_j$

In the coordinate system x'_i the components of are expressed as $L'_{ij} = \boldsymbol{e}'_i \cdot \boldsymbol{L}\boldsymbol{e}'_j$

We can easily evaluate the relation between the components L_{ij} and L'_{ij}. Using (1.20),

$$v'_i = y'_i - x'_i = c_{ij}(y_j - x_j) = c_{ij}v_j$$

$L'_{ij} = \boldsymbol{e}'_i \cdot \boldsymbol{L}\boldsymbol{e}'_j$ becomes $L'_{ij} = (c_{ik}\boldsymbol{e}_k) \cdot \boldsymbol{L}(c_{jl}\boldsymbol{e}_l) = c_{ik}c_{jl}\boldsymbol{e}_k \cdot \boldsymbol{L}\boldsymbol{e}_l = c_{ik}c_{jl}L_{kl}$ (1.52)

Recalling, $\dfrac{\partial x'_i}{\partial x_j} = c_{ij}$ $\quad \dfrac{\partial x_i}{\partial x'_j} = c_{ji}$ (1.21)

$$\Rightarrow L'_{ij} = \frac{\partial x'_i}{\partial x_k}\frac{\partial x'_j}{\partial x_l}L_{kl} \quad \text{or} \quad L'_{ij} = \frac{\partial x_k}{\partial x'_i}\frac{\partial x_l}{\partial x'_j}L_{kl}\,. \qquad (1.53)$$

In matrix notation, equation (1.52) is written as: $[L'] = [C][L][C]^T$

Tensor Algebra: transformation rule for Cartesian tensor

Definition of second order tensor: a matrix $[L]$ with nine components corresponds to a 2nd order tensor if its components are transformed according to (1.53):

$$L'_{ij} = \frac{\partial x'_i}{\partial x_k}\frac{\partial x'_j}{\partial x_l}L_{kl} \qquad \text{or} \qquad L'_{ij} = \frac{\partial x_k}{\partial x'_i}\frac{\partial x_l}{\partial x'_j}L_{kl}\,. \quad (1.53)$$

These transformation rules guarantee the invariance of $\boldsymbol{L}$ with respect to the choice of coordinates.

during a coordinate change that obeys (1.20):

$$v'_i = y'_i - x'_i = c_{ij}(y_j - x_j) = c_{ij}v_j \quad (1.20)$$

and that is an orthogonal transformation according to (1.16):

$$c_{ik}c^{-1}_{kj} = c_{ik}c_{jk} = \delta_{ij} \quad (1.16)$$

By generalizing these transformation rules, we can define a tensor of order *n*.
By definition, $\mathcal{T}$ is a tensor of order *n* if, during a coordinate transformation, its components aretransformed according to the rule:

$$\mathcal{T}'_{i_1 i_2 \cdots i_n} = \frac{\partial x'_{i_1}}{\partial x_{j_1}}\frac{\partial x'_{i_2}}{\partial x_{j_2}}\cdots\frac{\partial x'_{i_n}}{\partial x_{j_n}}\mathcal{T}_{j_1 j_2 \cdots j_n} \quad (1.55)$$

For a tensor of order 1 (vector) and order 2 we easily see:

$$v'_i = \frac{\partial x'_i}{\partial x_j}v_j \qquad L'_{ij} = \frac{\partial x'_i}{\partial x_k}\frac{\partial x'_j}{\partial x_l}L_{kl}$$

Tensor Algebra

Important rules of nth order tensors

Multiplication by a Scalar: the multiplication of a tensor of order *n* by a scalar is carried out by multiplying each component of the tensor by the scalar. The result is a tensor of order *n*.

Linear Combination: the linear combination of two tensors of order *n* is by linear combination of the corresponding components. A tensor of the same order is obtained.

Zero Tensor: it is the tensor for which all the components are equal to zero.

Equivalent Tensors: when the components of two tensors of the same order are equal term by term in a coordinate system, then they are equal in every other system; the tensors are equivalent.
Consequently, if a tensor relation is verified in one coordinate system, it is true in all coordinate systems.

Exterior Product of Tensors: Consider $A_{i1 \cdots in}$ *and* $B_{j1 \cdots jm}$ as the respective components of a tensor of order *n* and a tensor of order *m* in a coordinate system. The 3^{n+m} quantities obtained by:
$C_{i1 \cdots inj1 \cdots jm} = A_{i1 \cdots in} B_{j1 \cdots jm}$ form a tensor $\boldsymbol{C}$ of order $n + m$.
An example, is the dyadic product of two vectors (i.e. tensors of order 1) yields a tensor of order 2.

Tensor Algebra

Important rules of *n* order tensors

Tensor Contraction: Consider a tensor $\boldsymbol{A}$ of order *n* whose components in a coordinate system are $A_{i1 \cdots in}$.
The contraction consists of setting equal two indices of the tensor, i.e. the j^{th} and the k^{th} with j and $k \leq n$, and summing over these indices ($j,\ k = 1,\ 2,\ 3$) to form a tensor of order *n*-2 thus having 3^{n-2} components.
We say that this tensor is obtained by contraction of the indices *j* and *k*.
Example, L_{ii} is the only contraction possible of L_{ij} which is a scalar (tensor of order 0).

Consider two tensors $\boldsymbol{S}$ and $\boldsymbol{T}$ of order 2. Their exterior product results in a tensor $\mathcal{R}$ of order 4 with components:

$\mathcal{R}_{ijkl} = S_{ij}T_{kl}$. The components obtained by contraction of the second and third indices of $\mathcal{R}$ are:

$\mathcal{R}_{imml} = S_{im}T_{ml}$ .. We can show that this is a 2nd order 2.

From transformation (1.55), we can write:

$$\mathcal{R}'_{ijkl} = c_{ip}c_{jq}c_{kr}c_{ls}\mathcal{R}_{pqrs} \xrightarrow{\text{contraction}} \mathcal{R}'_{imml} = c_{ip}c_{mq}c_{mr}c_{ls}\mathcal{R}_{pqrs}$$

$$\xrightarrow{c_{ij}c_{qj} = \delta_{iq}} \boxed{\mathcal{R}'_{imml} = c_{ip}c_{ls}\,\delta_{qr}\mathcal{R}_{pqrs} = c_{ip}c_{ls}\mathcal{R}_{prrs} = \frac{\partial x'_i}{\partial x_p}\frac{\partial x'_l}{\partial x_s}\mathcal{R}_{prrs}}$$

i.e., from a 4th order to a 2nd order tensor

Tensor Algebra

(we consider tensors of order 2)

Sum of Tensors

Consider two tesnors $\boldsymbol{L}$ and $\boldsymbol{T}$. Their sum ($\boldsymbol{L}+\boldsymbol{T}$) is such that for every vector $\boldsymbol{a}$,

$$(\boldsymbol{T}+\boldsymbol{L})\boldsymbol{a} = \boldsymbol{T}\boldsymbol{a} + \boldsymbol{L}\boldsymbol{a}$$

with components:

$$(\boldsymbol{T}+\boldsymbol{L})_{ij} = \boldsymbol{e}_i \cdot (\boldsymbol{T}+\boldsymbol{L})\boldsymbol{e}_j = \boldsymbol{e}_i \cdot \boldsymbol{T}\boldsymbol{e}_j + \boldsymbol{e}_i \cdot \boldsymbol{L}\boldsymbol{e}_j \Rightarrow (\boldsymbol{T}+\boldsymbol{L})_{ij} = T_{ij} + L_{ij}$$

Interior Product of Two Tensors

Consider two tesnors $\boldsymbol{L}$ and $\boldsymbol{T}$. For or every vector $\boldsymbol{a}$, their products $\boldsymbol{LT}$ and $\boldsymbol{TL}$ are given by the equations:

$$(\boldsymbol{LT})\boldsymbol{a} = \boldsymbol{L}(\boldsymbol{T}\boldsymbol{a}) \quad \text{and} \quad (\boldsymbol{TL})\boldsymbol{a} = \boldsymbol{T}(\boldsymbol{L}\boldsymbol{a})$$

The components are:

$$\begin{aligned}(\boldsymbol{LT})_{ij} &= \boldsymbol{e}_i \cdot (\boldsymbol{LT})\boldsymbol{e}_j = \boldsymbol{e}_i \cdot \boldsymbol{L}(\boldsymbol{T}\boldsymbol{e}_j) \\ &= \boldsymbol{e}_i \cdot \boldsymbol{L}T_{mj}\boldsymbol{e}_m = T_{mj}\boldsymbol{e}_i \cdot \boldsymbol{L}\boldsymbol{e}_m\end{aligned}$$

In matrix form, the matrix of the interior product is equal to the product of the matrices of the two tensors:

$$[(LT)] = [L][T]$$

and

$$[(TL)] = [T][L]$$

Tensor Algebra

Interior Product of Two Tensors

Note that $\boldsymbol{LT} \neq \boldsymbol{TL}$ (not communicative).

For three tensors , $\boldsymbol{L}$, $\boldsymbol{T}$ and $\boldsymbol{S}$, we can write:

$$(\boldsymbol{L}(\boldsymbol{ST}))\boldsymbol{a} = (\boldsymbol{L}(\boldsymbol{ST})\boldsymbol{a}) = \boldsymbol{L}(\boldsymbol{S}(\boldsymbol{Ta}))$$

and $(\boldsymbol{LS})(\boldsymbol{Ta}) = \boldsymbol{L}(\boldsymbol{S}(\boldsymbol{Ta}))$

$\Rightarrow \boldsymbol{L}(\boldsymbol{ST}) = (\boldsymbol{LS})\boldsymbol{T}$ (it is associative)

When $\boldsymbol{L} = \boldsymbol{T}$

$\boldsymbol{TT} = \boldsymbol{T}^2$, $\boldsymbol{TT}^2 = \boldsymbol{T}^3$…..

Note also: $\det(\boldsymbol{ST}) = \det \boldsymbol{S} \det \boldsymbol{T}$

Useful relations of interior product:

$$\boldsymbol{L}(\boldsymbol{a} \otimes \boldsymbol{b}) = (\boldsymbol{La}) \otimes \boldsymbol{b}$$

$$(\boldsymbol{u} \otimes \boldsymbol{v})(\boldsymbol{a} \otimes \boldsymbol{b}) = (\boldsymbol{v} \cdot \boldsymbol{a})\boldsymbol{u} \otimes \boldsymbol{b} = \boldsymbol{u} \otimes \boldsymbol{b}(\boldsymbol{v} \cdot \boldsymbol{a})$$

Example 1.5

The first one is easily demonstrated as follows:

$$\begin{aligned}(\boldsymbol{L}(\boldsymbol{a} \otimes \boldsymbol{b}))_{ij} &= L_{im}(\boldsymbol{a} \otimes \boldsymbol{b})_{mj} = L_{im}a_m b_j \\ &= (\boldsymbol{La})_i b_j = ((\boldsymbol{La}) \otimes \boldsymbol{b})_{ij} \quad .\end{aligned}$$

Tensor Properties

Transpose of a Tensor:

It is obtained by exchanging two indices.
The transpose of L_{ij} is L_{ji} .

The transpose of the tesnor $\boldsymbol{L}$ is denoted as $\boldsymbol{L}^T$

and $\quad (\boldsymbol{L}^T)_{ij} = L_{ji}$

$$(\boldsymbol{LS})^T = \boldsymbol{S}^T \boldsymbol{L}^T$$

Also we can easily show that:

$$\boldsymbol{u} \cdot \boldsymbol{L}^T \boldsymbol{v} = \boldsymbol{Lu} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{Lu}$$

Note that:

$$\det \boldsymbol{L}^T = \det \boldsymbol{L}$$

Inverse of a Tensor:

For a tesnor with det $\boldsymbol{L} \neq 0$, there exist a unique tensor called inverse tensor and denoted $\boldsymbol{L}^{-1}$ of $\boldsymbol{L}$ and satisfies:

$$\boldsymbol{L}\boldsymbol{L}^{-1} = \boldsymbol{L}^{-1}\boldsymbol{L} = \boldsymbol{I}$$

We can easiliy show that:

$$(\boldsymbol{L}^{-1})^{-1} = \boldsymbol{L}$$

$$(\alpha \boldsymbol{L})^{-1} = \frac{1}{\alpha} \boldsymbol{L}^{-1}$$

$$\det(\boldsymbol{L}^{-1}) = (\det \boldsymbol{L})^{-1}$$

For two tensors:

$$(\boldsymbol{ST})^{-1} = \boldsymbol{T}^{-1}\boldsymbol{S}^{-1}$$

Tensor Properties

Symmetric Tensor:

It is defined when the tensor is equal to its transpose:

$$\boldsymbol{L} = \boldsymbol{L}^T \qquad \text{Or} \qquad L_{ij} = L_{ji}$$

Note that a symmetric tensor has 6 independent components.

antisymmetric Tensor:

A tensor $\boldsymbol{L}$ is said to be antisymmetric if

$$\boldsymbol{L} = -\boldsymbol{L}^T \qquad \text{Or} \qquad L_{ij} = -L_{ji}$$

It can be proven that all 2nd tensors can be uniquely decomposed into the sum of symmetric L^S and antisymmetric L^A tensors:

$$L_{ij} = L_{ij}^S + L_{ij}^A$$

Trace of a Tensor:

The trace of a 2nd order tensor $\boldsymbol{L}$, is the sum (scalar) of its diagonal elements and denoted by 'tr'

$$\mathrm{tr}\,(\boldsymbol{L}) = \mathrm{tr}\,\left(L_{ij}\,(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\right) =$$
$$= L_{ij}\;\mathrm{tr}\,(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) = L_{ij}\,\delta_{ij} = L_{ii}$$

For a dyad of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, it is their scalar product

$$\mathrm{tr}\,(\boldsymbol{a} \otimes \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b}$$

Some properties of the trace ($\alpha \in \mathbf{R}$)

$$\mathrm{tr}\,\boldsymbol{L} = \mathrm{tr}\,\boldsymbol{L}^T$$
$$\mathrm{tr}\,(\boldsymbol{S} + \boldsymbol{T}) = \mathrm{tr}\,\boldsymbol{S} + \mathrm{tr}\,\boldsymbol{T}$$
$$\mathrm{tr}\,(\alpha\boldsymbol{L}) = \alpha\;\mathrm{tr}\,\boldsymbol{L}$$
$$\mathrm{tr}\,(\boldsymbol{A}\boldsymbol{L}) = \mathrm{tr}\,(\boldsymbol{L}\boldsymbol{A})\,,$$

Tensor Properties

Deviatoric Tensor:

A tensor $\boldsymbol{L}$ can be decomposed into a spherical tensor $\boldsymbol{L}^S$ and a tensor with zero trace $\boldsymbol{L}^d$, called deviatoiric tensor, so that

$$\boldsymbol{L} = \boldsymbol{L}^s + \boldsymbol{L}^d$$

where

$$L^s_{ij} = \tfrac{1}{3} L_{kk} \delta_{ij}$$

$$L^d_{ij} = L_{ij} - \frac{1}{3} \operatorname{tr}(\boldsymbol{L})\, \delta_{ij}$$

Note that the components of the deviatoric tensor are not independent.

Orthogonal Tensor:

For a tesnor $\boldsymbol{Q}$, that satisfies the condition:

$$\boldsymbol{Q}\boldsymbol{u} \cdot \boldsymbol{Q}\boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v} \qquad \text{for every } \boldsymbol{u} \text{ and } \boldsymbol{v}.$$

Using relation $\boldsymbol{u} \cdot \boldsymbol{L}^T \boldsymbol{v} = \boldsymbol{L}\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{L}\boldsymbol{u}$
we obtain:

$$\boldsymbol{u} \cdot \boldsymbol{Q}^T \boldsymbol{Q} \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v}$$

➡ An orthogonal tensor satisfies $\boldsymbol{Q}^T \boldsymbol{Q} = \boldsymbol{I}$

Note that since $\boldsymbol{u} \cdot \boldsymbol{v}$ is preserved, the angle between the vectors as well as their norms $\|\boldsymbol{u}\|$, $\|\boldsymbol{v}\|$ are preserved.

Note that the matrix of $\boldsymbol{Q}$ is equal to the matrix $[C]$ of a rotation of the basis vectors.

Tensor Properties

Scalar Product of two tensors:

For two tensors $\boldsymbol{S}, \boldsymbol{T}$ of odrer two the following scalar is defined as scalar product:

$$a = S_{ij}T_{ij} = \boldsymbol{S} : \boldsymbol{T}$$

It is a double tensor contraction.

The norm of a tensor $\boldsymbol{L}$ is defined as:

$$\|\boldsymbol{L}\| = (\boldsymbol{L} : \boldsymbol{L})^{1/2} = (L_{ij}L_{ij})^{1/2} \geq 0$$

Properties of scalar product:

$$\boldsymbol{L} : (\boldsymbol{ST}) = (\boldsymbol{S}^T\boldsymbol{L}) : \boldsymbol{T} = (\boldsymbol{LT}^T) : \boldsymbol{S}$$

$$(\boldsymbol{u} \otimes \boldsymbol{v}) : (\boldsymbol{a} \otimes \boldsymbol{b}) = (\boldsymbol{u} \cdot \boldsymbol{a})(\boldsymbol{v} \cdot \boldsymbol{b})$$

$$\boldsymbol{L} : (\boldsymbol{a} \otimes \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{Lb} = (\boldsymbol{a} \otimes \boldsymbol{b}) : \boldsymbol{L}\,.$$

Example 1.6:

Verify the last identity. Following the definition we obtain:

$$(\boldsymbol{L} : (\boldsymbol{a} \otimes \boldsymbol{b})) = L_{ij}(\boldsymbol{a} \otimes \boldsymbol{b})_{ij} =$$

$$= L_{ij}a_ib_j = a_iL_{ij}b_j = (\boldsymbol{a} \cdot \boldsymbol{Lb})$$

Example 1.7:

Consider two tensors $\boldsymbol{A}, \boldsymbol{B}$ such that $A_{ij} = A_{ji}$ (symmetric) and $B_{ij} = -B_{ji}$ (antisymmetric). Show that their scalar product is zero (use the properties of symmetric and antisymmetric tensors).

$$(\boldsymbol{A} : \boldsymbol{B}) = A_{ij}B_{ij} = \frac{1}{2}(A_{ij}B_{ij} + A_{ij}B_{ij})$$

$$= \frac{1}{2}(A_{ij}B_{ij} - A_{ij}B_{ji}) = \frac{1}{2}(A_{ij}B_{ij} - A_{ji}B_{ij})$$

$$= \frac{1}{2}(A_{ij}B_{ij} - A_{ij}B_{ij}) = 0$$

Dual Vector of a 2nd order tensor

The dual vector components d_i of a tensot $\boldsymbol{L}$ are defined by the product:

$$d_i = \frac{1}{2}\varepsilon_{ikj}L_{jk} = -\frac{1}{2}\varepsilon_{ijk}L_{jk}$$

Explicitly they are:

$$d_1 = -\frac{1}{2}(\varepsilon_{123}L_{23} + \varepsilon_{132}L_{32}) = -\frac{1}{2}(L_{23} - L_{32})$$
$$d_2 = -\frac{1}{2}(\varepsilon_{231}L_{31} + \varepsilon_{213}L_{13}) = -\frac{1}{2}(L_{31} - L_{13})$$
$$d_3 = -\frac{1}{2}(\varepsilon_{312}L_{12} + \varepsilon_{321}L_{21}) = -\frac{1}{2}(L_{12} - L_{21})$$

The dual vector has zero components if $\boldsymbol{L}$ is symmetric:

$$(L_{ij} = L_{ji})$$

This is shown as follows. From the definition of the vector and spliting the tensor in two parts we can write:

$$d_i = -\frac{1}{2}(\varepsilon_{ijk}L^S_{jk} + \varepsilon_{ijk}L^A_{jk})$$

The first term in the parentehsis is zero (product of symmetric and antisymmetric tensors) and the antisymmetric part is zero due to symmetry.

For any tensor $\boldsymbol{L}$ (using the definiton) the dual vector depends on the antisymmetric part:

$$d_i = -\frac{1}{2}\varepsilon_{ijk}L^A_{jk}$$

Further it can be shown that: →

$$d_1 = -L^A_{23}$$
$$d_2 = -L^A_{31}$$
$$d_3 = -L^A_{12}$$

Eigenvalues and Eigenvectors of a tensor

For a tensot $\boldsymbol{L}$ if $\boldsymbol{u}$ is a vector that when $\boldsymbol{L}$ is applied is transformed into avector parralell to itself:

$$\boldsymbol{L}\boldsymbol{u} = \lambda \boldsymbol{u}$$

Then $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{L}$ and λ is the corresponding **eigenvalue**. Conventionally the eigenvectors are normalized to vectors $\boldsymbol{n}$ of unit length (unit **eigenvector**):

$$\Rightarrow \boldsymbol{L}\boldsymbol{n} = \lambda \boldsymbol{n} = \lambda \boldsymbol{I}\boldsymbol{n}$$

Or $(\boldsymbol{L} - \lambda \boldsymbol{I})\boldsymbol{n} = \boldsymbol{0}$ with $\boldsymbol{n} \cdot \boldsymbol{n} = 1$

$$\downarrow \quad \boldsymbol{n} = n_i \boldsymbol{e}_i$$

$$(L_{ij} - \lambda \delta_{ij}) n_j = 0 \quad n_j n_j = 1$$

$$\xrightarrow{\boldsymbol{n} \neq \boldsymbol{0}} \det([L] - \lambda[I])$$

The last equation is called **characteristic equation** of the tesnor $\boldsymbol{L}$. Its solution gives the eigenvalues and eigenvectors.

For a symmetric tensor $(L_{ij} = L_{ji})$ the following theorem is holds for its matrix (from linear algebra):

Theorem: The eigenvalues of a real *nxn* symmetric matrix are all real. The corresponding eigenvectors are orthogonal.

The solution of the characteristic equations results in the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ & corresponding eigenvectors $\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3$.

Note that

1: If $\lambda_1 \neq \lambda_2$ then $\boldsymbol{n}_1 . \boldsymbol{n}_2 = 0$ and $\boldsymbol{n}_1, \boldsymbol{n}_2$ are orthogonal.

2: If $\lambda_1 = \lambda_2 \neq \lambda_3$ we have $\boldsymbol{n}_1 . \boldsymbol{n}_3 = \boldsymbol{n}_2 . \boldsymbol{n}_3 = 0$. In such case directions $\boldsymbol{n}_1, \boldsymbol{n}_2$, are chose mutually orthogonal and normal to $\boldsymbol{n}_3$.

3: $\lambda_1 = \lambda_2 = \lambda_3$ the directions $\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3$ are chosen mutually orthogonal and without restriction.

Eigenvalues and Eigenvectors of a tensor

The characteristic equation:

$$(L_{ij} - \lambda\,\delta_{ij})n_j = 0$$

is a third order polynomial:

$$\lambda^3 - I_1(\boldsymbol{L})\lambda^2 + I_2(\boldsymbol{L})\lambda - I_3(\boldsymbol{L}) = 0$$

Its solution gives the eigenvalues and eigenvectors:

The following three parameters are called invariants of the tensor ***L*:**

$$I_1(\boldsymbol{L}) = L_{ii} = \mathrm{tr}\,\boldsymbol{L}$$

$$\begin{aligned} I_2(\boldsymbol{L}) &= \begin{vmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{vmatrix} + \begin{vmatrix} L_{22} & L_{23} \\ L_{32} & L_{33} \end{vmatrix} + \begin{vmatrix} L_{11} & L_{13} \\ L_{31} & L_{33} \end{vmatrix} \\ &= \frac{1}{2}\left(L_{ii}L_{jj} - L_{ij}L_{ji}\right) \\ &= \frac{1}{2}\left((\mathrm{tr}\ \boldsymbol{L})^2 - \mathrm{tr}\,(\boldsymbol{L}\boldsymbol{L})\right) = \frac{1}{2}\left((\mathrm{tr}\ \boldsymbol{L})^2 - \mathrm{tr}\,(\boldsymbol{L}^2)\right) \end{aligned}$$

$$I_3(\boldsymbol{L}) = \varepsilon_{ijk}L_{i1}L_{j2}L_{k3} = \det\boldsymbol{L}\,.$$

Note that any independent combination of these invariants results in another invariant.

<u>Example 1.9</u>

The expression $a = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{ist}L_{js}L_{kt}$

is an invariant of the tensor ***L*.**

Use the identity

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

to modify the expression as follows:

$$\begin{aligned} 2a &= \varepsilon_{ijk}\varepsilon_{ist}L_{js}L_{kt} = \left(\delta_{js}\delta_{kt} - \delta_{jt}\delta_{ks}\right)L_{js}L_{kt} \\ &= \delta_{js}L_{js}\delta_{kt}L_{kt} - \delta_{jt}L_{js}\delta_{ks}L_{kt} \\ &= L_{jj}L_{kk} - L_{ts}L_{st} = L_{jj}L_{kk} - L_{ts}L_{ts}\ . \end{aligned}$$

Eigenvalues and Eigenvectors of a tensor

Positive definite tensor

Such a tensor satifies the following relation

$$\forall \boldsymbol{v} \in E^3, \quad \boldsymbol{v} \cdot \boldsymbol{L}\boldsymbol{v} > 0$$

It can be shown that the eigenvalues of a positive Definite tensor are all positive:

For the tensor $\boldsymbol{L}$ with one of its eigenvalue λ and and corresponding eigenvector $\boldsymbol{n}$, we can easily see that since

$$\boldsymbol{L}\boldsymbol{n} = \lambda \boldsymbol{n} \Rightarrow \boldsymbol{n} \cdot \boldsymbol{L}\boldsymbol{n} = \lambda > 0$$

Spectral decomposition of a tensor or spectral representation of a tensor

For a tensor $\boldsymbol{L}$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3$, and corresponding eigenvectors $\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3$.
The orthogonal eigenvectors form a basis for the spectral decompostion writen as follows:

$$\boldsymbol{L} = \sum_{i=1}^{3} \lambda_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i$$

We can show it as follows:

use $\boldsymbol{L} = \boldsymbol{L}\boldsymbol{I}$ and $\boldsymbol{I} = \boldsymbol{n}_i \otimes \boldsymbol{n}_i$

$$\boldsymbol{L} = \boldsymbol{L}(\boldsymbol{n}_i \otimes \boldsymbol{n}_i) = (\boldsymbol{L}\boldsymbol{n}_i) \otimes \boldsymbol{n}_i = (\lambda_i \boldsymbol{n}_i) \otimes \boldsymbol{n}_i = \sum_{1}^{3} \lambda_i (\boldsymbol{n}_i \otimes \boldsymbol{n}_i)$$

$$\boldsymbol{L}(\boldsymbol{a} \otimes \boldsymbol{b}) = (\boldsymbol{L}\boldsymbol{a}) \otimes \boldsymbol{b} \quad (1.65)$$

$$\boldsymbol{L}\boldsymbol{n} = \lambda \boldsymbol{n} \quad (1.109)$$

Square root of a tensor & Polar decomposition

<u>Theorem (square root)</u>

For a symmetric ,positiove definite tensor $\boldsymbol{C}$ with eigenvalues λ_i^2 and corresponding eigenvectors $\boldsymbol{n}_i$, there is a symmetric positive definite tensor $\boldsymbol{U}$ such that:

$$\boldsymbol{U}^2 = \boldsymbol{C}$$

and denote it as $\sqrt{\boldsymbol{C}} = \boldsymbol{U}$

These two tensors have the following spectral forms:

$$\boldsymbol{C} = \sum_{i=1}^{3} \lambda_i^2 \, \boldsymbol{n}_i \otimes \boldsymbol{n}_i$$

$$\boldsymbol{U} = \sum_{i=1}^{3} \lambda_i \, \boldsymbol{n}_i \otimes \boldsymbol{n}_i$$

<u>Theorem</u> (polar decomposition)

For a tensor $\boldsymbol{F}$ with with determinant det $\boldsymbol{F} > 0$ there exist symmetric positive definite tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ and a rotation (an orthogonal tensor with a positive Determinant equal to 1) $\boldsymbol{R}$ such that:

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}$$

These decompositions are unique and we have:

$$\boldsymbol{U} = \sqrt{\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}} \quad \text{and} \quad \boldsymbol{V} = \sqrt{\boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}}$$

Representation $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$ is called right decomposition.
Representation $\boldsymbol{F} = \boldsymbol{V}\boldsymbol{R}$ is called left decomposition.

Functions of a tensor

Isotropic tensor function of a symmetric tensor

By definition an tensor isotropic function $\boldsymbol{f}$, for which the variable is a 2nd order symmetric tensor $\boldsymbol{T}$, satisfies the identity:

$$\boldsymbol{Q}\boldsymbol{f}(\boldsymbol{T})\boldsymbol{Q}^T = \boldsymbol{f}(\boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T)$$

for any orthogonal tensor $\boldsymbol{Q}$.

For a symmetric tensor $\boldsymbol{L}$ the following relation is true:

$$\boldsymbol{L} = \boldsymbol{f}(\boldsymbol{T})$$

Rivlin-Ericksen Theorem

The last expression can be written in the form

$$\boldsymbol{L} = \varphi_0(I_1(\boldsymbol{T}), I_2(\boldsymbol{T}), I_3(\boldsymbol{T}))\boldsymbol{I} + \varphi_1(I_1(\boldsymbol{T}), I_2(\boldsymbol{T}), I_3(\boldsymbol{T}))\boldsymbol{T} + \varphi_2(I_1(\boldsymbol{T}), I_2(\boldsymbol{T}), I_3(\boldsymbol{T}))\boldsymbol{T}^2,$$

φ_i $(i = 0,\ 1,\ 2)$ *are scalar functions of the invariants of* $\boldsymbol{T}$

Scalar function of a tensor

The function $\mathcal{W}(\boldsymbol{T})$ is defined as a scalar funtion of the tensor $\boldsymbol{T}$ and yield a scalar. When $\boldsymbol{T}$ is symmetric and the condition:

$$\mathcal{W}(\boldsymbol{T}) = \mathcal{W}(\boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T)$$

is satisfied, then $\mathcal{W}(\boldsymbol{T})$ is isotropic of $\boldsymbol{T}$ and is represenetd by:

$$\mathcal{W}(\boldsymbol{T}) = \Phi(I_1(\boldsymbol{T}), I_2(\boldsymbol{T}), I_3(\boldsymbol{T}))$$

where the parameters $I_1(\boldsymbol{T}), I_2(\boldsymbol{T}), I_3(\boldsymbol{T})$ are the invariants of $\boldsymbol{T}$. This is also equivalent to:

$$\mathcal{W}(\boldsymbol{T}) = \phi(\lambda_1, \lambda_2, \lambda_3)$$

where λ_1, λ_2, λ_3, are the eirenvalues of $\boldsymbol{T}$.

It can be shown that for the isotropic function $\mathcal{W}(\boldsymbol{T})$ its derivative with respct to $\boldsymbol{T}$ is:

$$\frac{\partial \mathcal{W}}{\partial \boldsymbol{T}} = \sum_{i=1}^{3} \frac{\partial \mathcal{W}}{\partial \lambda_i} \boldsymbol{n}_i \otimes \boldsymbol{n}_i$$

Tensor Analysis

Notation:

Scalar field

$F(x_i,\ t)$ → $F(x_1, x_2, x_3, t)$

Vector field

$v_i\ (x_m, t)$: it covers all three components

$v_1(x_1, x_2, x_3, t)$, $v_2\ (x_1, x_2, x_3, t)$, $v_3\ (x_1, x_2, x_3, t)$.

Tensor field

$L_{ij}\ (x_m, t)$: it covers all nine components

$L_{11}\ (x_1, x_2, x_3, t)$, $L_{12}\ (x_1, x_2, x_3, t)$, …., $L_{33}\ (x_1, x_2, x_3, t)$,

Derivatives:

For a tensor $\boldsymbol{L} = \boldsymbol{L}(t)$ the derivative with respect to a scalar parameter (i.e., time) is a tensor of the same order

$$\dot{\boldsymbol{L}} = \frac{d\boldsymbol{L}}{dt} = \lim_{\Delta t \to 0} \frac{\boldsymbol{L}(t + \Delta t) - \boldsymbol{L}(t)}{\Delta t}$$

In terms of its components, it is given by

$$\dot{\boldsymbol{L}} = \frac{dL_{ij}(t)}{dt} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \dot{L}_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

For a vector $\boldsymbol{v}(t)$ the first and second time derivatives are

$$\frac{d\boldsymbol{v}}{dt} = \dot{\boldsymbol{v}} = \dot{v}_i(t)\boldsymbol{e}_i \quad \text{and} \quad \frac{d^2\boldsymbol{v}}{dt^2} = \ddot{\boldsymbol{v}} = \ddot{v}_i(t)\boldsymbol{e}_i$$

Tensor Analysis

Derivatives:

The following identites are establied easily

$$\frac{d}{dt}(\boldsymbol{u} \pm \boldsymbol{v}) = \frac{d\boldsymbol{u}}{dt} \pm \frac{d\boldsymbol{v}}{dt}$$
$$\frac{d}{dt}(\boldsymbol{u} \otimes \boldsymbol{v}) = \frac{d\boldsymbol{u}}{dt} \otimes \boldsymbol{v} + \boldsymbol{u} \otimes \frac{d\boldsymbol{v}}{dt}$$
$$\frac{d}{dt}(\boldsymbol{L} \pm \boldsymbol{T}) = \frac{d\boldsymbol{L}}{dt} \pm \frac{d\boldsymbol{T}}{dt}$$
$$\frac{d}{dt}(\alpha(t)\boldsymbol{L}) = \frac{d\alpha(t)}{dt}\boldsymbol{L} + \alpha(t)\frac{d\boldsymbol{L}}{dt}$$

$$\frac{d}{dt}(\boldsymbol{L}\boldsymbol{T}) = \frac{d\boldsymbol{L}}{dt}\boldsymbol{T} + \boldsymbol{L}\frac{d\boldsymbol{T}}{dt}$$
$$\frac{d}{dt}(\boldsymbol{L}\boldsymbol{a}) = \frac{d\boldsymbol{L}}{dt}\boldsymbol{a} + \boldsymbol{L}\frac{d\boldsymbol{a}}{dt}$$
$$\frac{d}{dt}(\boldsymbol{L}^T) = \left(\frac{d\boldsymbol{L}}{dt}\right)^T .$$

Derivatives

Demonstration of $\frac{d}{dt}(\boldsymbol{L}\boldsymbol{a}) = \frac{d\boldsymbol{L}}{dt}\boldsymbol{a} + \boldsymbol{L}\frac{d\boldsymbol{a}}{dt}$

From the defintion $\dot{\boldsymbol{L}} = \frac{d\boldsymbol{L}}{dt} = \lim_{\Delta t \to 0} \frac{\boldsymbol{L}(t+\Delta t) - \boldsymbol{L}(t)}{\Delta t}$

$$\begin{aligned}
\frac{d}{dt}(\boldsymbol{L}\boldsymbol{a}) &= \lim_{\Delta t \to 0} \frac{\boldsymbol{L}(t+\Delta t)\boldsymbol{a}(t+\Delta t) - \boldsymbol{L}(t)\boldsymbol{a}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big(\boldsymbol{L}(t+\Delta t)\boldsymbol{a}(t+\Delta t) - \boldsymbol{L}(t)\boldsymbol{a}(t) \\
&\qquad + \boldsymbol{L}(t)\boldsymbol{a}(t+\Delta t) - \boldsymbol{L}(t)\boldsymbol{a}(t+\Delta t)\Big) \\
&= \lim_{\Delta t \to 0} \frac{\big(\boldsymbol{L}(t+\Delta t) - \boldsymbol{L}(t)\big)\boldsymbol{a}(t+\Delta t)}{\Delta t} \\
&\qquad + \lim_{\Delta t \to 0} \frac{\boldsymbol{L}(t)\big(\boldsymbol{a}(t+\Delta t) - \boldsymbol{a}(t)\big)}{\Delta t} \\
&= \frac{d\boldsymbol{L}}{dt}\boldsymbol{a} + \boldsymbol{L}\frac{d\boldsymbol{a}}{dt} .
\end{aligned}$$

Tensor Analysis

Gradient of a scalar field

Associated with a scalar field $F(\boldsymbol{x})$ is a vector field called the gradient of F.

It is denoted $\boldsymbol{\nabla} F$ or $\mathbf{grad}\, F$ and is such that the scalar product with $d\boldsymbol{x}$ gives the difference between the values of F evaluated at $\boldsymbol{x} + d\boldsymbol{x}$ and at $\boldsymbol{x}$ we obtain

$$dF = F(\boldsymbol{x} + d\boldsymbol{x}) - F(\boldsymbol{x}) = \boldsymbol{\nabla} F \cdot d\boldsymbol{x}.$$

With $\boldsymbol{e} = d\boldsymbol{x}/dx$ $(dx = \|d\boldsymbol{x}\|)$

$$\left(\frac{dF}{dx}\right)_{\text{in direction } \boldsymbol{e}} = \boldsymbol{\nabla} F \cdot \boldsymbol{e}$$

In Cartesian coordinates we have:

$$\boldsymbol{\nabla} F = \frac{\partial F}{\partial x_1}\boldsymbol{e}_1 + \frac{\partial F}{\partial x_2}\boldsymbol{e}_2 + \frac{\partial F}{\partial x_3}\boldsymbol{e}_3 = \frac{\partial F}{\partial x_i}\boldsymbol{e}_i$$

Gradient of a vector field

With a vector field $\boldsymbol{v}(\boldsymbol{x})$, we associate a tensor, called the gradient of $\boldsymbol{v}$, and denote it $\boldsymbol{\nabla v}$. It is a tensor of order 2 which, applied to $d\boldsymbol{x}$, gives the difference of $\boldsymbol{v}$ between $\boldsymbol{x} + d\boldsymbol{x}$ and $\boldsymbol{x}$.

We have $d\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x} + d\boldsymbol{x}) - \boldsymbol{v}(\boldsymbol{x}) = (\boldsymbol{\nabla v})\, d\boldsymbol{x}$

With $\boldsymbol{e} = d\boldsymbol{x}/dx$ $(dx = \|d\boldsymbol{x}\|)$

We obtain $$\left(\frac{d\boldsymbol{v}}{dx}\right)_{\text{in direction } \boldsymbol{e}} = (\boldsymbol{\nabla v})\,\boldsymbol{e}$$

in direction 1 $$\left(\frac{d\boldsymbol{v}}{dx}\right)_{\text{in direction } \boldsymbol{e}_1} = \frac{\partial \boldsymbol{v}}{\partial x_1} = (\boldsymbol{\nabla v})\,\boldsymbol{e}_1$$

$$(\boldsymbol{\nabla v})_{11} = \boldsymbol{e}_1 \cdot (\boldsymbol{\nabla v})\,\boldsymbol{e}_1 = \boldsymbol{e}_1 \cdot \frac{\partial \boldsymbol{v}}{\partial x_1} = \frac{\partial}{\partial x_1}(\boldsymbol{e}_1 \cdot \boldsymbol{v}) = \frac{\partial v_1}{\partial x_1}$$

$$(\boldsymbol{\nabla v})_{ij} = \frac{\partial v_i}{\partial x_j}$$

Tensor Analysis

Gradient of a scalar valued tensor function

For a regular, scalar valued, smooth function $\mathcal{W}(\boldsymbol{T})$ of a tensor $\boldsymbol{T}$ of order 2, the first two terms of a Taylor series expansion around $\boldsymbol{T}$ are:

$$\mathcal{W}(\boldsymbol{T}+d\boldsymbol{T}) = \mathcal{W}(\boldsymbol{T}) + d\mathcal{W}(\boldsymbol{T}) + \mathrm{o}(d\boldsymbol{T})$$

Where $\mathrm{o}(d\boldsymbol{T})$ the remainder of the expansion which tends to zero as $d\boldsymbol{T}$ → 0, as expressed in the relation

$$\lim_{d\boldsymbol{T}\to 0} \frac{\mathrm{o}(d\boldsymbol{T})}{\|d\boldsymbol{T}\|} = 0$$

The total differential is expressed as follows

$$d\mathcal{W}(\boldsymbol{T}) = \frac{\partial \mathcal{W}(\boldsymbol{T})}{\partial \boldsymbol{T}} : d\boldsymbol{T} = \mathrm{tr}\left(\left(\frac{\partial \mathcal{W}(\boldsymbol{T})}{\partial \boldsymbol{T}}\right)^T d\boldsymbol{T}\right)$$

Or
$$d\mathcal{W}(\boldsymbol{T}) = \frac{\partial \mathcal{W}(\boldsymbol{T})}{\partial T_{ij}} dT_{ij}$$

2nd order tensor defined as gradient $\mathcal{W}(\boldsymbol{T})$ in $\boldsymbol{T}$

Gradient of a tensor valued tensor function

For a regular, tensor valued, smooth function $\boldsymbol{S}(\boldsymbol{T})$ a tensor $\boldsymbol{T}$ of order 2, the first two terms of a Taylor expansion around $\boldsymbol{T}$ are:

$$\boldsymbol{S}(\boldsymbol{T}+d\boldsymbol{T}) = \boldsymbol{S}(\boldsymbol{T}) + d\boldsymbol{S}(\boldsymbol{T}) + o(d\boldsymbol{T})$$

When $d\boldsymbol{T} \to 0$ we have:

$$d\boldsymbol{S}(\boldsymbol{T}) = \frac{\partial \boldsymbol{S}(\boldsymbol{T})}{\partial \boldsymbol{T}} : d\boldsymbol{T}.$$

Or $$dS_{ij} = \frac{\partial S_{ij}}{\partial T_{kl}} dT_{kl}$$

The tensor $\partial \boldsymbol{S}(\boldsymbol{T})/\partial \boldsymbol{T}$ is of order 4 and is the gradient of $\boldsymbol{S}(\boldsymbol{T})$ in $\boldsymbol{T}$.

Tensor Analysis

Divergence of vectors and tensors

Let $\boldsymbol{v}(\boldsymbol{x})$ be a vector field. The divergence of $\boldsymbol{v}(\boldsymbol{x})$ is the scalar obtained by a contraction:

$$\operatorname{div} \boldsymbol{v} = \frac{\partial v_i}{\partial x_i} = \operatorname{tr}(\boldsymbol{\nabla} \boldsymbol{v}) \quad \text{or} \quad \boldsymbol{\nabla} \cdot \boldsymbol{v} = \frac{\partial v_i}{\partial x_i}$$

When the divergence of a vector field $\boldsymbol{v}(\boldsymbol{x})$ is zero, that is, $\operatorname{div} \boldsymbol{v} = 0$ the field $\boldsymbol{v}(\boldsymbol{x})$ is called a **solenoidal** field.

For a tensor $\boldsymbol{L}$ we have its divergence $\operatorname{div} \boldsymbol{L}$ defined as a vector:

$$(\operatorname{div} \boldsymbol{L})_i = \frac{\partial L_{ij}}{\partial x_j} = L_{ij,j}$$

Or

$$\operatorname{div} \boldsymbol{L} = \frac{\partial L_{ik}}{\partial x_j} (\boldsymbol{e}_i \otimes \boldsymbol{e}_k) \boldsymbol{e}_j = \frac{\partial L_{ik}}{\partial x_j} (\boldsymbol{e}_k \cdot \boldsymbol{e}_j) \boldsymbol{e}_i$$

$$= \frac{\partial L_{ik}}{\partial x_j} \delta_{kj} \boldsymbol{e}_i = \frac{\partial L_{ij}}{\partial x_j} \boldsymbol{e}_i \, .$$

Curl of a vector field

For a vector field $\boldsymbol{v}(\boldsymbol{x})$ the curl is defined as a vector:

$$\textbf{curl}\, \boldsymbol{v} = \boldsymbol{\nabla} \times \boldsymbol{v}$$

In index notation it is $(\textbf{curl}\, \boldsymbol{v})_i = \varepsilon_{ijk} \frac{\partial v_k}{\partial x_j}$

Using the property of the permutation symbol we get its three components:

$$(\textbf{curl}\, \boldsymbol{v})_1 = \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}$$

$$(\textbf{curl}\, \boldsymbol{v})_2 = \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}$$

$$(\textbf{curl}\, \boldsymbol{v})_3 = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$$

If the curl of the field $\boldsymbol{v}(\boldsymbol{x})$ is zero, that is,

$$\boldsymbol{\nabla} \times \boldsymbol{v} = 0$$

The field is called **irrotational.**

Tensor Analysis

Laplacian of a scalar field

We also encounter second order derivatives in expressions of physical quantities in mechanics. The divergence of the gradient of a scalar function is an example:

$$\frac{\partial^2 F}{\partial x_i \partial x_i} \qquad \text{or} \qquad \nabla \cdot (\boldsymbol{\nabla} F) \text{ or } \operatorname{div}(\textbf{grad}\, F)$$

which is also the Laplacian of F denoted $\nabla^2 F$ or $\triangle F$:

$$\frac{\partial^2 F}{\partial x_i \partial x_i} = \frac{\partial^2 F}{\partial {x_1}^2} + \frac{\partial^2 F}{\partial {x_2}^2} + \frac{\partial^2 F}{\partial {x_3}^2}$$

When $\nabla^2 F = 0$ the function is said harmonic known as Laplace's equation.

When $\nabla^2 F = f$ where f is scalar it is called Poisson's equation.

Laplacian of a vector field

We can also treat a vector function in the same way. The divergence of the gradient of a vector is written as:

$$\frac{\partial^2 v_j}{\partial x_i \partial x_i} \qquad \text{or} \qquad \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \boldsymbol{v}) \text{ or } \textbf{div}\,(\boldsymbol{\nabla} \boldsymbol{v})$$

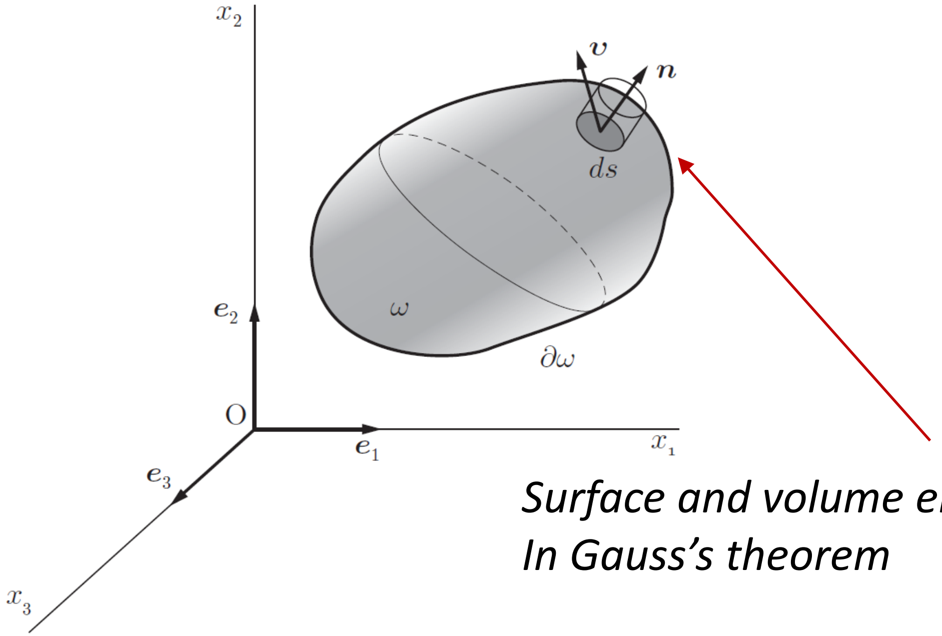
The result of these operations is a vector. We also denote the operation as ∇^2, that is,

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \boldsymbol{v}) = \nabla^2 \boldsymbol{v}\,.$$

Tensor Analysis

Definition of the flux

Consider a body in a 3D space of volume ω and surface $\partial\omega$:



Surface and volume elements In Gauss's theorem

The flux of a property Q through the surface of the body is ($\boldsymbol{v}$ is interpreted as the velocity field):

$$\int_{\partial\omega} Q v_i n_i \, ds \qquad \text{or} \qquad \int_{\partial\omega} Q(\boldsymbol{v}\cdot\boldsymbol{n})\, ds$$

Gauss theorem

The Gauss theorem or divergence theorem, written as follows for an arbitrary component $T_{jk.....}(x_i)$:

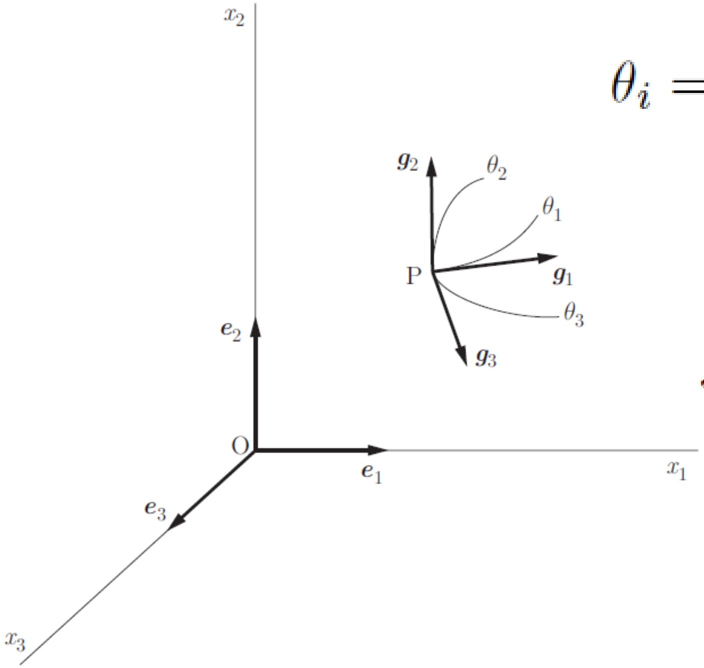
$$\int_{\omega} \frac{\partial T_{jk.....}}{\partial x_i}\, dv = \int_{\partial\omega} n_i T_{jk.....}\, ds$$

This theorem transforms the volume integral of the divergence of a property of a continuous medium into a surface integral and plays an important role in mechanics of continuous media.

Curvilinear coordinates

Consider a point P with Cartesian coordinate x_i .
In a curvilinear coordinate system θ_i
(<u>cylindrical or spherical coordinates</u>), the position of the point P is given by the three numbers θ_i that represent the coordinate curves passing through P that is, by the curves on which two of the three coordinates θ_i are constant.
The curvilinear coordinates can be considered as functions of Cartesian coordinates:

$$\theta_i = \theta_i(x_j)$$

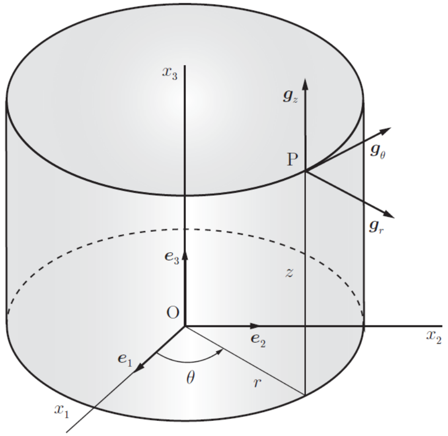


If the jacobian:

$$J = \det(\partial\theta_i/\partial x_j)$$

is not zero the transformation Is invertible.

Cylindrical coordinates



$$\theta_1 = r = \sqrt{x_1^2 + x_2^2}$$

$$\theta_2 = \theta = \tan^{-1}\frac{x_2}{x_1}$$

$$\theta_3 = z = x_3$$

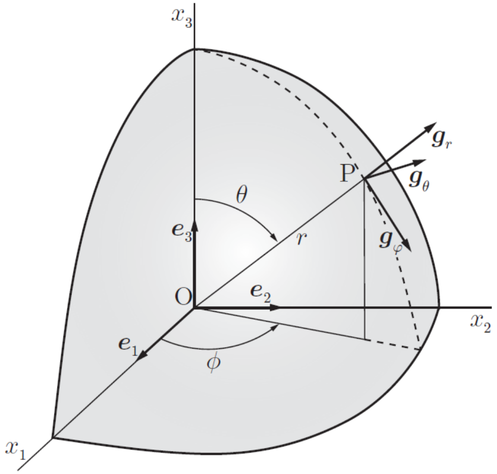
$$x_1 = r\cos\theta$$

$$x_2 = r\sin\theta$$

$$x_3 = z\,.$$

$$dV = r\,dr\,d\theta\,dz.$$

Spherical coordinates



$$\theta_1 = r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\theta_2 = \theta = \tan^{-1}\frac{\sqrt{x_1^2 + x_2^2}}{x_3}$$

$$\theta_3 = \varphi = \tan^{-1}\frac{x_2}{x_1}\,.$$

$$x_1 = r\sin\theta\cos\varphi$$

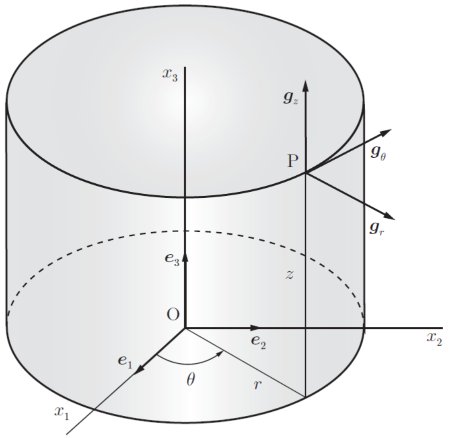
$$x_2 = r\sin\theta\sin\varphi$$

$$x_3 = r\cos\theta\,.$$

$$dV = r^2\sin\theta\,dr\,d\theta\,d\varphi.$$

Curvilinear coordinates

Cylindrical coordinates



$$\theta_1 = r = \sqrt{x_1^2 + x_2^2}$$

$$\theta_2 = \theta = \tan^{-1}\frac{x_2}{x_1}$$

$$\theta_3 = z = x_3$$

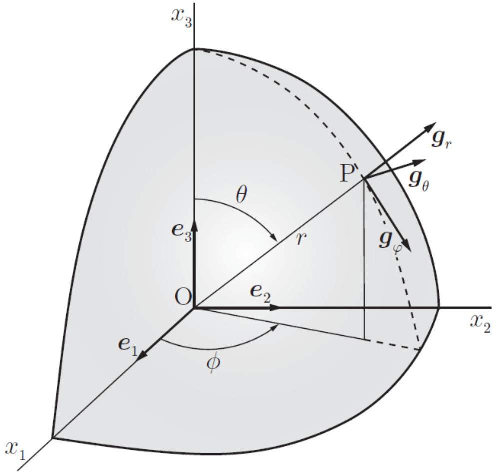
$$x_1 = r\cos\theta$$

$$x_2 = r\sin\theta$$

$$x_3 = z\,.$$

$$dV = r\,dr\,d\theta\,dz.$$

Spherical coordinates



$$\theta_1 = r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\theta_2 = \theta = \tan^{-1}\frac{\sqrt{x_1^2 + x_2^2}}{x_3}$$

$$\theta_3 = \varphi = \tan^{-1}\frac{x_2}{x_1}\,.$$

$$x_1 = r\sin\theta\cos\varphi$$

$$x_2 = r\sin\theta\sin\varphi$$

$$x_3 = r\cos\theta\,.$$

$$dV = r^2\sin\theta\,dr\,d\theta\,d\varphi.$$

All relevant parameters and operators are found in:

Appendix A: Cylindrical coordinates
Appendix B: Spherical Coordinates

In the book: Mechanics of Continuous Media (Botsis and Deville PPUR 2018)